Pergamon

*Topology* Vol. 34. No. 2. pp. 473–476. 1995
Elsevier Science Ltd
Printed in Great Britain
0040-9383/95 \$9.50 + 0.00

# A PROOF OF PRZYTYCKI'S CONJECTURE ON $n$-RELATOR 3-MANIFOLDS

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(*Received* 18 *May* 1992; *in revised form* 27 *July* 1993)

## §0. INTRODUCTION

IN 1983, Przytycki [5] first gave sufficient conditions for a 1-relator 3-manifold with incompressible boundary, and in 1984, Jaco [3] extended Przytycki's result by using a geometric approach. But there exist examples (see [5]) which indicate that a direct generalization of above results to the case of $n$-relator 3-manifolds is not possible. In 1987, Przytycki [6] proposed a set of conditions which he conjectured to be sufficient for an $n$-relator 3-manifold with incompressible boundary, that is,

*Przytycki's Conjecture.* Let $C = \{J_1, \ldots, J_n\}$ be a family of 2-sided, pairwise disjoint, simple closed curves in the boundary of a handlebody $H$ (with genus $k > 0$). Assume the following conditions are satisfied (see §1 for necessary definitions):

(0) $\partial H - C$ is incompressible in $H$,
(1) for each $j$, $\partial H - (C - J_j)$ is compressible in $H$ (or equivalently, the family of elements of $\pi_1(H) = F_k$ represented by $C - J_j$ does not bind the free group $F_k$),
(2) for each pair $j$, $s (j \neq s)$, $C - \{J_j, J_s\}$ does not bind any free factor $F_{k-1}$ of $F_k = F_{k-1} \times F_1$,

(p) no $(n-p)$-element subfamily of $C$ binds a free factor $F_{k-p+1}$ of $F_k = F_{k-p+1} \times F_{p-1}$,

$(n-1)$ no curve $J_j$ of $C$ binds a free factor $F_{k-n+2}$ of $F_k = F_{k-n+2} \times F_{n-2}$.

Then the $n$-relator 3-manifold $H_c$ has incompressible boundary, or it is equal to $D^3$.

In [2] Przytycki proved his conjecture true for $n = 1$, 2 and 3. When $n > 3$, he had examples showing that all the assumptions in his conjecture are necessary.

The main result of this paper is that we give an answer to Przytycki's conjecture affirmatively.

## §1. PRELIMINARIES

We work in the PL-category and use s.c.c. as an abbreviation of simple closed curve (or curves).

*Definition* 1.1. Let $M$ be a 3-manifold and $S$ a surface which is either properly embedded in $M$ or contained in $\partial M$. We say that $S$ is compressible (in $M$) if one of the following conditions is satisfied:

(1) $S$ is a 2-sphere which bounds a 3-cell in $M$, or
(2) $S$ is a 2-cell and either $S \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset S \cup \partial M$, or

(3) there is a 2-cell $D \subset M$ with $D \cap S = \partial D$ and $\partial D$ is not contractible in $S$.

In the case (3), $D$ is also called a compressing disk for $S$ (in $M$).

We say that $S$ is incompressible if each component of $S$ is not compressible.

*Definition* 1.2. Let $M$ be a 3-manifold and $J$ a 2-sided s.c.c. on $\partial M$. Let $A_J$ be a regular neighbourhood of $J$ in $\partial M$, $(D^3, A)$ a 3-cell with an annulus $A \subset \partial D^3$, and $h$: $A_J \to A$ a homeomorphism. Then the 3-manifold $(M, A_J)\bigcup_h(D^3, A)$ is denoted by $M_J$. If $C = \{J_1, \ldots, J_n\}$ is a collection of pairwise disjoint, 2-sided s.c.c. on $\partial M$, then we denote $M_C = (\ldots((M_{J_1})_{J_2})\ldots)_{J_n}$. In particular, when $M$ is a handlebody $H$ with genus $k(>0)$, $H_C$ is called an $n$-relator 3-manifold.

Clearly, the definition of $M_c$ does not depend on the order of the $J_i$.

*Definition* 1.3. Let $C = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint 2-sided s.c.c. on a surface $S$. We say that a s.c.c. $J \subset S - C$ is coplanar with $C$ if $J$ cuts a disk with holes from $S$ cut open along $C$ (i.e. $S - C$).

*Definition* 1.4. Let $W \subset F_k$ be a set of cyclic words in the free group $F_k$ with a basis $X$. The incidence graph $J(W)$ is the graph whose vertices are in 1-1 correspondence with the non-trivial words in $W$, with an edge joining vertices $w_1$ and $w_2$ if there exists $x \in X$ such that $x$ or $x^{-1}$ lies in $w_1$ and $x$ or $x^{-1}$ lies in $w_2$. $W$ is connected with respect to the basis $X$ if $J(W)$ is connected, and is connected if it is connected with respect to each basis of $F_k$. If the set $W$ of cyclic elements is not contained in any proper free factor of $F_k$ and if $W$ is connected, we say that $W$ binds $F_k$.

For convenience, we shall refer to disks with holes as "planar surfaces". We shall also abuse notation slightly by using the symbol $C$, which represents a family of closed curves in the 3-manifold $M$, also to represent the corresponding elements of conjugacy classes of $\pi_1(M)$ when this causes no confusion.

The following two lemmas will be used in our proof.

LEMMA 1.1. (Due to Jaco [3]) *Let $M$ be a 3-manifold with compressible boundary and $J$ a 2-sided s.c.c. in $\partial M$. If $\partial M - J$ is incompressible in $M$ then $\partial M_J$ is incompressible in $M_J$ or $M_J$ is equal to $D^3$.*

Observe that Lemma 1.1 is valid for a non-compact 3-manifold, so as an immediate consequence we get

LEMMA 1.2. (Lemma 2.3 in [2]) *Let $J_1, \ldots, J_n (n > 1)$ be a family of 2-sided, pairwise disjoint s.c.c. on the boundary of a 3-manifold $M$. Let $\partial M - \bigcup_{i=1}^{n} J_i$ be incompressible in $M$ and $\partial M - \bigcup_{i=1}^{n-1} J_i$ compressible in $M$. Then $\partial M_{J_n} - \bigcup_{i=1}^{n-1} J_i$ is incompressible in $M_{J_n}$.*

## §2. THEOREM AND ITS PROOF

THEOREM. Przytycki's conjecture is true.

*Proof.* The theorem was proved for $n = 1$, 2 and 3 by Przytycki (see [5] and [6]). We only need to consider the case of $n > 3$.

Let $\mathscr{X}_i$ denoted the set which consists of all the $i$-element subsets of $C = \{J_j, 1 \leqslant j \leqslant n\}$, $1 \leqslant i \leqslant n$, and for $K \in \mathscr{X}_i$, write $\bar{K} = C - K$, thus $\bar{K} \in \mathscr{X}_{n-i}$.

We know, by the assumption conditions (0), (1) and Lemma 1.2, that $\partial H_{J_i} - \{J_j, i \leqslant j \leqslant n, j \neq i\}$ is incompressible in $H_{J_i}$, for $i = 1, 2, \ldots, n$. That is for any $K \in \mathscr{X}_1$, $\partial H_K - \bar{K}$ is incompressible in $H_K$.

Now we suppose that for some $m < n$ $(m \geqslant 1)$ and each $i \leqslant m$, $\partial H_K - \bar{K}$ is incompressible in $H_K$ for any $K \in \mathscr{X}_i$. We shall prove that for any $K \in \mathscr{X}_{m+1}$, $\partial H_K - \bar{K}$ is incompressible in $H_K$. Therefore the proof of the theorem will be finished by a finite induction on $m$.

Let $K \in \mathscr{X}_{m+1}$, say, $K = \{J_1, \ldots, J_{m+1}\}$. Write $L = K - J_{m+1}$, then $L \in \mathscr{X}_m$. From the assumption we know that $\partial H_L - \bar{L}$ is incompressible in $H_L$ and we need $\partial H_L - \bar{K}$ to be compressible (then we can use Lemma 1.2 if $m < n - 1$, or Lemma 1.1 if $m = n - 1$). By the assumption (1), $\partial H - (L \cup \bar{K})$ is compressible in $H$. We divide it into two cases to discuss:

(i) there exists a disk $D$ of compression of $\partial H - (L \cup \bar{K})$ with $\partial D$ not coplanar with $L$. Then $\partial D$ is not trivial in $\partial H_K - \bar{K}$, so $\partial H_K - \bar{K}$ is compressible in $H_L$, thus Lemma 1.2 or 1.1 implies that $\partial H_K - \bar{K}$ is incompressible in $H_K$ or $\bar{K} = \varnothing$ and $H_K = D^3$.

(ii) for each compressing disk $D$ of $\partial H - (L \cup \bar{K})$ in $H$, $\partial D$ is coplanar with $L$ in $\partial H$. In this situation we have:

*Assertion.* There exists a compressing disk $D$ of $\partial H - (L \cup \bar{K})$ which does not separate $H$.

In fact, let $D_1$ be a compressing disk of $\partial H - (L \cup \bar{K})$. If $D_1$ does not separate $H$, the Assertion already holds. If $D_1$ separates $H$, then $H = H_1 \triangle H_2$ (with genus of $H_i > 0$ for $i = 1, 2$), where $\triangle$ denotes the boundary connected sum realized by $D_1$. Since $\partial D_1$ is coplanar with $L$, so $\partial D_1$ cuts out of a planar surface $S$ from $\partial H$ cut open along $L$. Let $J_i^+$ and $J_i^-$ denote the two cutting sections of $J_i$ for $J_i \in L$, then $\partial S - \partial D_1$ is contained in the cutting section set of a minimal subset $L_1$ of $L$ (that is, $\partial S - \partial D_1$ is not contained in the cutting section set of any proper subset of $L_1$). With losing no generality, say, $L_1 = \{J_{p+1}, J_{p+2}, \ldots, J_m\}$ $(0 \leqslant p < m)$. Then no curve $J_j$ of $C - L_1 - \{J_{m+1}\}$ is in $S$ (certainly $C - L_1 - \{J_{m+1}\}) \cap \partial S = \varnothing$). If the opposite happens, say, $J_j$ of $C - L_1 - \{J_{m+1}\} = K_1$ is in $S$, then $J_j$ is trivial in $\partial H_{L_1}$, so $\partial H_{L_1} - \bar{L}$ is compressible in $H_{L_1}$, this contradicts the inductive assumption. We also have that $\partial S - \partial D_1$ are all paired (i.e. if $J_j^+$ or $J_i^- \in \partial S$ then $J_j^-$ or $J_j^+ \in \partial S$) because if $J_j^+$ is in $\partial S$ but $J_j^-$ is not, then $J_j$ is coplanar with $\partial D_1 \cup L_2$ where $L_2 = L_1 - J_j$. Then $J_j$ is trivial in $H_{L_2}$ so $\partial H_{L_2} - \bar{L}_2$ is compressible in $H_{L_2}$ which again contradicts the inductive assumption. Thus we can assume that $L_1 \subset \partial H_2 (g(H_2) = m - p)$ and $K_1 \subset \partial H_1 (g(H_1) = k - m + p)$.

By the assumption conditions, $K_1$ does not bind $F_{k-m+p} = \pi(H_1)$, so $\partial H_1 - K_1$ is compressible, and each compressing disk of $\partial H_1 - K_1$ in $H_1$ is a compressing disk of $\partial H - (L \cup \bar{K})$ so its boundary is coplanar with $L$ (in fact, a proper subset of $L$). If $D_2$ is a compressing disk of $\partial H_1 - K_1$ and $D_2$ does not separate $H_1$, then $D_2$ does not separate $H$, the proof of the Assertion is done. Otherwise, after repeating a finite number of the same steps (if necessary) we can reduce the situation to that:

$$H = H' \triangle H'', \quad g(H') = k - m, \quad g(H'') = m, \quad L \subset \partial H'', \quad \bar{K} \subset \partial H'.$$

By the assumption conditions, $\bar{K}$ does not bind the free factor $F_{k-m} = \pi_1(H')$ of $F_k$, so $\partial H' - \bar{K}$ is compressible and each compressing disk of $\partial H' - \bar{K}$ is a compressing disk of $\partial H - (L \cup \bar{K})$ in $H$, therefore its boundary is coplanar with $L$ in $\partial H$. But this is impossible because $\bar{K} \subset H'$. This contradiction shows our Assertion.

Let $D$ be a compressing disk of $\partial H - (L \cup \bar{K})$ which does not separate $H$. Since $\partial D$ is coplanar with $L$, so $\partial D$ and a subset $A$ (with at least two elements, by the assumption (0)) of the cutting set of $L$ bound a planar surface $S$ in $\partial H$. Because $D$ does not separate $H$, there exists an element, say $J_m^+$, in $A$ such that $J_m^-$ is not in $A$. If opposite, $\partial D$ separates $\partial H$, and this is a contradiction. Write $L_2 = L - \{J_m\}$, then by the property of $S$ we know that $J_m$ is trivial in $H_{L_2}$, so $\partial H_{L_2} - \bar{L}_2$ is compressible. But this contradicts the inductive assumption. This contradiction implies, in fact, that the case (ii) can not occur.

This finishes the proof of the theorem.

*Remark.* From the proof we know that our method is valid for $n = 1, 2$, and 3.

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